Statistical Evaluation and Processing of Uncorrelated and Correlated Outputs of the SPOrt Radiometer

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Abstract. The outputs of SPOrt radiometer present different statistical properties. The determination of the statistics of SPOrt radiometer outputs allows the determination of the maximum likelihood estimator for the best possible estimation of the signals. In the data processing 1/f instrumental noise can be eliminated through the technique of circulant matrices.

INTRODUCTION

The SPOrt Experiment has been conceived to obtain a measurement of the polarized component of the sky emissions at microwaves. The measurement of this component has a great importance for scientists and a successful experiment would be an important achievement for radio-astronomical research.

The SPOrt radiometer will provide two different outputs: the total power measurements and the Q and U Stokes parameter measurements. Observations shall be performed at 22, 32, 60 and 90GHz.

In our work the input signal, corresponding to the Cosmic Background Radiation (CBR), is represented as white noise. It means that this kind of signal can be characterized through a normal distribution.

At the end of the Ortho Mode Transducer (OMT), on each channel of the instrument there will be a signal with these characteristics. However, it must be remarked that the signals on the two radiometer channels are not statistically independent because of the presence of polarized CBR. The determination of output signal statistics has been a significant achievement of the SPOrt radiometer analysis.

DETERMINATION OF INSTRUMENT SENSITIVITY

By appropriately defining the functional blocks of the instrument, both for the total power outputs and the Q and U Stokes parameters' outputs, it is possible to define the sensitivity of the instrument for the two different outputs.

The sensitivity is defined by determining the minimum signal power that can be detected by the instrument (*the system noise output power is equal to the output power due to the sky signals*).

By expressing it in terms of equivalent temperature, the sensitivity will be:

Total power
$$\Delta T = T_{sys} \cdot \sqrt{\frac{1}{B \cdot \tau}} \qquad (1)$$

Q and U Stokes parameters $$\Delta T = T_{sys} \cdot \sqrt{\frac{1}{2 \cdot B \cdot \tau}} \qquad (2)$$

T_{sys} is the system noise temperature, B is the pre-detection bandwidth and τ is the time constant of post-detection low-pass filter. In the equation it has been assumed that no channel gain instability occurs.

STATISTICS OF RADIOMETER OUTPUTS

The outputs of SPOrt radiometer present different statistical properties. The determination of the statistics of SPOrt radiometer outputs is necessary because it allows the determination of the maximum likelihood estimator that will allow the best possible estimation of the signals.

After the band-pass filtering each signal can be represented as a complex narrow-band Gaussian process with zero mean value. The generic signal of this kind will be indicated with x(t). It will be:

$$x(t) = u(t) + j \cdot v(t) \qquad (3)$$

where the signal components u(t) and v(t) are real Gaussian processes. The probability density function (PDF) of x(t) is a complex Gaussian PDF. It can be obtained by assuming that u(t) and v(t) are statistically independent. It is supposed that they are real Gaussian processes with null mean and equal variance (corresponding to the half part of the x(t) variance σ_x^2).

For the total power measurement square law devices are employed for signal detection. The detector output can be defined as follows:

$$y(t) = a \cdot |x(t)|^2 = a \cdot \left[u^2(t) + v^2(t)\right] \qquad (4)$$

In the expression, a is the constant parameter characteristic of the square law device.

With the appropriate mathematical steps the PDF of y has been evaluated. The signal can be expressed as the sum of two terms, corresponding to the square of u and v. It will be:

$$y = w + z \qquad (5)$$

With the appropriate mathematical steps, the PDF of y is obtained.

$$f_y(y) = \int_{-\infty}^{+\infty} f_w(y-z) \cdot f_z(z) \cdot dz = \frac{1}{a \cdot \sigma_x^2} \exp\left(-\frac{y}{a \cdot \sigma_x^2}\right) \qquad (6)$$

In the total power case the density function is exponential. Through its knowledge it is possible to evaluate the mean value of y, representing the total power of the input signal entering a channel. However, the total power measurement does not allow the discrimination of the polarized CBR power from the unpolarised term contribution.

This result can be achieved by evaluating the Q and U Stokes parameters. The corresponding output of the radiometer can be expressed in the following form:

$$y = x_A \cdot x_B^* \qquad (7)$$

The PDF of the random process defined by previous equation can be evaluated by expressing the signals in terms of their real and imaginary parts and by introducing the multivariate complex Gaussian PDF.

$$\mathbf{x} = \begin{pmatrix} x_A \\ x_B \end{pmatrix} \tag{8}$$

The expression of the multivariate complex Gaussian PDF is as follows:

$$f_{\mathbf{x}}(\mathbf{x}) = \frac{1}{\pi^2 \det(\mathbf{C_x})} \exp\left[-(\mathbf{x}-\boldsymbol{\mu})^H \cdot \mathbf{C_x}^{-1} \cdot (\mathbf{x}-\boldsymbol{\mu})\right] \tag{9}$$

The vector $\boldsymbol{\mu}$ consists of the mean values of the complex processes that compose **x**. According to the previous assumptions, the two mean values will be null in the present case.

The matrix $\mathbf{C_x}$ is the covariance matrix of **x**.

$$f_{x_A,x_B}(x_A, x_B) = \frac{1}{\pi^2 \sigma_x^4 \cdot (1-|r|^2)} \exp\left[-\frac{|x_A|^2 + |x_B|^2 - 2\Re\{r^* x_A x_B^*\}}{\sigma_x^2 \cdot (1-|r|^2)}\right] \tag{10}$$

The term r is the cross-correlation coefficient between the signals on the two channels. It is complex and has a module lower than 1. With the appropriate mathematical steps, the final expression of the PDF is obtained. It is not an exponential PDF as in the total power output case and has a more complex expression.

$$f_y(y) = \frac{2}{\pi \sigma_x^4 \cdot (1-|r|^2)} \cdot \exp\left[\frac{2 \cdot \Re\{r^* \cdot y\}}{\sigma_x^2 \cdot (1-|r|^2)}\right] \cdot K_0\left[\frac{2|y|}{\sigma_x^2 \cdot (1-|r|^2)}\right] \tag{11}$$

The function K_0 is the modified Bessel function of the second type, of 0 order.

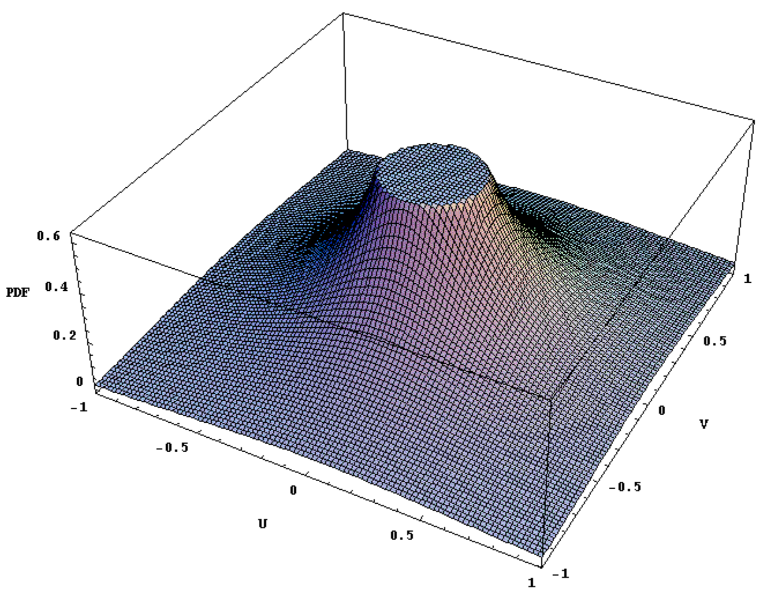

FIGURE 1. PDF of y as a function of u and v, with r amplitude equal to 0.1 and phase equal to 30^o

Applying the necessary analytical elaboration, the resulting expression of mean value of y is the following:

$$E\{y\} = r \cdot \sigma_x^2 \tag{12}$$

In Figure 1 the PDF of y for r with amplitude 0.1 and phase 30^{o} has been represented. Variance of x has been assumed equal to 1.

ESTIMATORS

Through the previously determined density functions, it is possible to verify if arithmetical the average of the measurements represents the best possible estimator.

For the measurement of the total power output, the analysis has demonstrated that the arithmetical average is the Minimum Variance Unbiased Estimator (MVUB) for this kind of data. This conclusion depends on the PDF of the radiometer output corresponding to total power measurements, determined as an exponential density function.

However, due to the complexity of PDF it is quite probable that this estimator will not allow the maximum likelihood estimate of Stokes parameters.

Following these considerations, it can be concluded that the arithmetical average is not the MVUB estimator for Q and U Stokes parameters. Assuming that the main goal of the SPOrt experiment is the determination of Q and U, a different estimator should be chosen.

POST-PROCESSING OF MEASURED DATA

Another important topic in SPOrt Experiment is the post-processing of CBR data.

Sky mapping is affected by the long-term effects of 1/f instrumental noise and by the spurious polarisation effects due to the real characteristics of the radiometer.

The removal of the effects of spurious polarisation is based on the data properties depending on orbit characteristics. For the elimination of instrument noise it is possible to develop appropriate signal processing algorithms that allow the determination of the cosmic radiofrequency signal.

The results of the measurements can be represented as M signal levels grouped in a column vector **y**, that is the TOD (Time Ordered Data) vector:

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ .. \\ y_M \end{bmatrix} \tag{13}$$

It can be expressed as follows:

$$\mathbf{y}(t) = \mathbf{x}(t) + \mathbf{n}(t) = \mathbf{x}(t) + \mathbf{n}'(t) + \mathbf{n}''(t) \tag{14}$$

where **n**'(t) is the white noise contribution and **n**''(t) is the 1/f noise contribution. The output signal vector **x**(t) is related to the vector $\boldsymbol{\theta}$ of the sky pixels CBR contributions through the known matrix **H** that depends on the antenna pattern:

$$\mathbf{x} = \mathbf{H} \cdot \boldsymbol{\theta} \tag{15}$$

The aim of signal processing technique employment is the estimation of $\boldsymbol{\theta}$. The measured data are to be processed with Data Adaptive Rank-Shaping Methods, so that it is possible to evaluate the signal contributions.

Assuming the employment of linear methods, the estimator vector may be expressed as:

$$\tilde{\boldsymbol{\theta}} = \mathbf{W} \cdot \mathbf{y} = \left[\mathbf{H}^{T} \cdot \mathbf{M} \cdot \mathbf{H}\right]^{-1} \mathbf{H}^{T} \cdot \mathbf{M} \tag{16}$$

The matrix $\mathbf{M}$ is the inverse of the covariance matrix of the noise vector $\mathbf{n}$, indicated as $\mathbf{N}$. The removal of 1/f noise can be performed assuming that $\mathbf{N}$ is circulant and symmetric. A circulant and symmetric matrix has the following generic form:

$$\mathbf{N} = \begin{pmatrix} c_0 & c_1 & c_2 & .. & c_2 & c_1 \\ c_1 & c_0 & c_1 & .. & 0 & c_2 \\ c_2 & c_1 & c_0 & .. & 0 & 0 \\ .. & & & & & .. \\ c_2 & 0 & 0 & & c_0 & c_1 \\ c_1 & c_2 & 0 & .. & c_1 & c_0 \end{pmatrix} \tag{17}$$

The filtering of 1/f noise is performed by a matrix based on the results of the decomposition of $\mathbf{N}$ (Fourier matrix and eigenvalues matrix).

The estimation of $\boldsymbol{\theta}$ has been performed with reduced rank. Different estimators have been considered, depending on the weight coefficients (Abrupt, Unbiased, Maximum Likelihood, Conditional Mean and Wiener estimation).

Software tools in MATLAB[1] have been employed for simulating the previously mentioned steps. The Mean Square Error (MSE) has been evaluated for each kind of estimator depending on the Signal to Noise Ratio (SNR) and assuming different values for both the signal mean value and its standard deviation. It is representative of the estimator effectiveness in achieving the knowledge of signals to be determined.

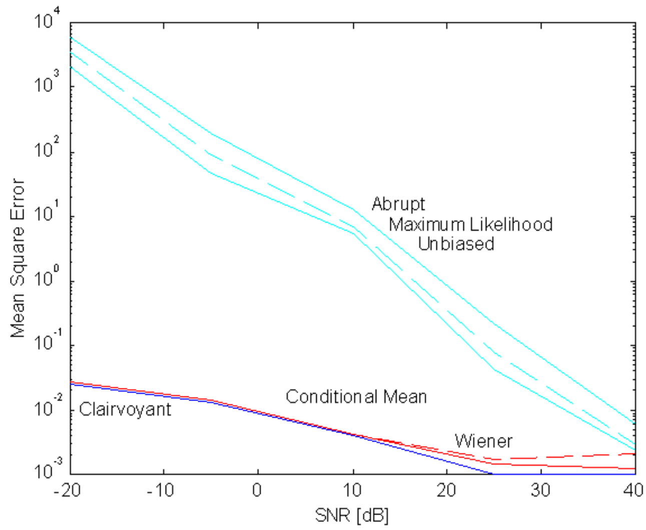

FIGURE 2. MSE for different estimators (Mean Value=0, Standard Deviation=0.1).

[1] MATLAB is the software package for mathematical and technical computing by TheMathWorks, Inc.

In the following diagrams, several cases have been reported. Wiener estimator can be employed for null mean value signals only.

The simulation results show that the Mean Square Error is minimum for a very low Signal to Noise Ratio with the Conditional Mean estimator, while the performance of the other estimators (Abrupt, Maximum Likelihood, Unbiased) significantly improves for high SNR levels. However, in SPOrt Experiment very low SNR levels are to be expected, so the Conditional Mean estimation is the best for the filtering of output measurements.

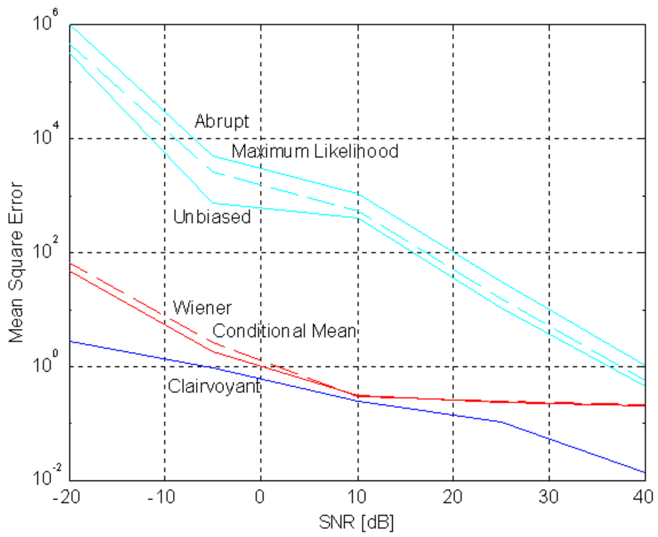

FIGURE 3. MSE for different estimators (Mean Value=0, Standard Deviation=1)

CONCLUSION

The previously reported analysis has shown that the typical considerations about output characteristics of total power radiometers can be applied to SPOrt Experiment too. However, the outputs of Q and U Stokes parameters' measurements present different statistical properties. The involvement of appropriate signal processing techniques may be an useful support for the removal of noise contributions and the cleaning of data.